Symmetric Embedding of Locally Regular Hyperbolic Tilings

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Abstract

Hyperbolic tilings of type {3,7}, {7,3}, {4,5}, and {5,4} are mapped onto closed 2-manifolds of genus 3 through 7 with as much symmetry as possible. All these maps exhibit local regularity which makes all vertices look equivalent, but most lack some of the global flag-transitive symmetries that are topologically possible. On the other hand, many of these mappings allow some nice symmetrical embeddings in 3D Euclidean space with tiles of limited deformation, which makes it possible to create attractive Escher-like tiling patterns on these surfaces of higher genus.

1. Introduction

The goal of this study is to find "regular" hyperbolic tilings with as many "identical" tiles as possible on surfaces ranging from genus 3 through genus 7. In this context "identical" can be understood in two different ways. In a geometrical interpretation we are looking for tiles that are geometrically congruent, so that we would have to make a minimum number of different molds if we wanted to cast them. But there is also a topological interpretation. In that case we assume that the tiles are made from some highly stretchable material and are free to move around on the underlying smooth surface of appropriate genus; they might thus all have different shapes, but we still insist that they all have the same topological neighborhoods. "Regular" also has two interpretations. We definitely want all the tilings to be "locally regular," meaning that all tiles are *p*-gons with the same *p*, joining in vertices that all have the same valence *v*. A more ambitious goal is to also make these tilings "globally regular," meaning that they have flag-transitive symmetry, so that every flag(vertex, edge, face) is topologically identical to every other such flag. Locally regular tilings are primarily of artistic interest; they lend themselves well to the construction of Escher-like tiling patterns. Globally regular tilings are exciting to mathematicians interested in group theory. To illustrate these concepts we now discuss some examples of {3,7} tilings on a genus-3 surface (Fig.1).

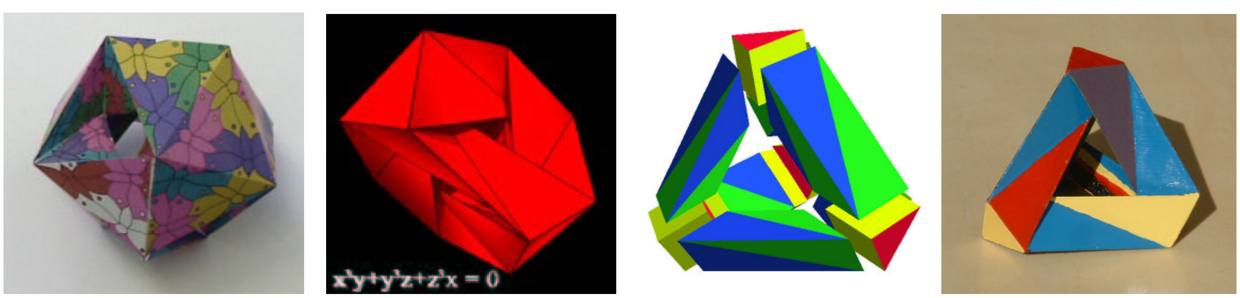

Figure 1: *Regular {3,7} tilings on a genus-3 surface: (a) locally regular tiling by Douglas Dunham; (b) globally regular map by Schulte and Wills; (c) generator for other {3,7} tilings by using fractional Dehn twists on the prismatic arms of the tetroid surface; (d) globally regular map generated this way.*

At the Bridges conference in 2002 Douglas Dunham presented a delightful way of covering a genus-3 polyhedron with 56 isosceles triangles with 168 butterflies [3]. The underlying polyhedron (Fig.1a) is formed by two nested icosahedra (which here should be seen, more appropriately, as "snub tetrahedra") of opposite handedness, in which the four primary faces of each have been eliminated, and then act as portals for four 3-sided anti-prism tunnels, each connecting the inner and outer shells. This mesh of 56 triangles constitutes a locally regular tiling, in which exactly 7 triangles meet at each of the 24 vertices. The beauty is that this map can be realized as an embedded polyhedron in 3-dimensional space with well-formed triangles, in which the ratio of longest-to-shortest edge in any triangle is at most about 8:5. This polyhedron is thus far less "twisty" than the structure (Fig.1b) described by Schulte and Wills [12], which also is a regular {3,7} tiling with 56 triangles on a genus-3 surface. However, this latter structure has another attractive property. It has the full combinatorial symmetry group of order 168 of Klein's famous "quartic curve" [7]. It thus forms a globally regular, flag-transitive map with 24×7=168 orientation-preserving automorphisms. This is the largest possible conformal symmetry group for a tiled Riemann surface of genus 3 [6]. It has been celebrated by Helaman Ferguson with his marble sculpture "The Eight-fold Way" [4] located at the Mathematical Science Research Institute in Berkeley (MSRI), and a whole book has been written about it [10] to celebrate its inauguration.

It turns out that both of the above triangle meshes by Dunham and by Schulte et al. can emerge as variants of the same underlying constructive procedure: Place four triangular prisms at the corners (hubs) of a tetrahedral frame, and realize the six edges of this frame as anti-prisms that connect two 4-sided faces on two adjacent hub prisms. The hub prisms can be placed with maximal symmetry with respect to the tetrahedral frame; then the six arms take on rectangular cross sections, and pairs of anti-prism faces become coplanar (Fig.1c). By changing the hubs also into anti-prisms and twisting their inner and outer equilateral triangle faces against one another, chiral asymmetry is introduced. One can also make these inner and outer hub triangles of different sizes, so as to render the triangles in the connecting arms less skinny. Overall, there are three degrees of freedom that can be adjusted, while still maintaining the 12-fold rotational symmetries of the tetrahedron: 1.) The relative size of the two equilateral triangles at the four hubs. 2.) Their relative distances from the center of the tetrahedral frame. 3.) The twist angle around the common 3-fold symmetry axes applied in opposite directions to the inner and outer triangles at the hubs. With suitable adjustment of these three parameters, Dunham's polyhedron can be obtained as one of many possible shapes with orientation-preserving tetrahedral symmetry.

Our generator can produce many locally regular tilings with different global connectivity, if the anti-prism arms of the tetroid surface are given different amounts of twist. If we twist one of the arms through a full 360°, we end up with the same local connections from which we started, and the resulting mesh has the same connectivity as the original one, but it has a different embedding in 3D space. Such a 360° twist is called a Dehn twist [2], and it can be applied along any closed cut around an arm or around one or more holes of a surface of non-zero genus. Now, for our locally regular tilings we can also apply fractional Dehn twists that twist an arm by an integral fraction of 360°, given by the number of tiles clustered around that arm – or by an integer multiple of that value. Such fractional twists still maintain the valances of the vertices and the local regularity of the tiling, but they change the global connectivity of the mesh. Specifically for the tetroid configurations shown in Figures 1c and 1d, every arm has four distinct values of possible fractional Dehn twists. The combination of different twists on different arms allows us to generate many different meshes with different connectivity. But all these combinations still amount to only a small fraction of all possible locally regular {3,7} tilings on a genus-3 surface – of which there are 11'277 according to the table titled "equivelar surfaces" in [8]. Only two combinations of fractional Dehn twist – one quarter clockwise from our starting position (Fig.1c), and its mirror image – will result in a map that is globally regular and has the full combinatorial symmetry of Klein's quartic curve.

In order to obtain some of the many other regular tilings, we may start with a surface of genus 3 that exhibits a different Euclidean symmetry group, and adapt the tiling on its surface accordingly. Our approach is a generalization of the generator introduced in Figure 1c. To obtain more geometrical freedom in embedding our surfaces in 3D Euclidean space, we generalize our anti-prism arms into flexible anti-prism tubing, thus allowing us to construct tubular surfaces with fewer junctions (or "hubs"). We will discuss this approach in some detail for genus-3 surfaces. The same method can then readily be used to make locally regular tilings on surfaces of higher genus.

2. Locally Regular Tilings on Surfaces of Genus 3

We construct various genus-3 surfaces by seamlessly gluing together generalized cylinders in *m*-way junctions, which all should have the same valence *m*, so that we can obtain maximal symmetry. To obtain the discrete triangular facets needed to form a locally regular map of 56 triangular facets and 24 valence-7 vertices, we tessellate the arms into *s*-sided anti-prisms, which are glued onto appropriate Archimedean polyhedra to form the necessary *m*-way junctions. Moreover, the construction described above, using anti-prism arms glued onto select faces of Archimedean solids, will result directly in the desired meshing with all valence-7 vertices. Thus for the genus-3 surface we have the following options:

Using four 3-way junctions and six 4-sided anti-prism arms (Fig.2a): This is just a curved variant of Figure 1c. The underlying junction bodies are 3-sided prisms. After 3 arms have been glued to the quadrilateral faces, only the two triangular (black) end-faces remain exposed. Thus the resulting surface has the following facets: 4*2 equilateral junction triangles and 6*8 arm triangles = 56 total.

Using two 4-way tetrahedral junctions and four 6-sided anti-prism arms (Fig.2b): The junction bodies are truncated tetrahedra. After 4 arms have been glued to the hexagon faces, the four triangular (black) truncation faces remain exposed. Thus the resulting surface has the following facets: 2*4 equilateral junction triangles and 4*12 arm triangles = 56 total.

Using one 6-way cuboid junction and three 8-sided anti-prism arm loops (Fig.2c): The junction body is a truncated cube. After the arm loops have been glued to the octagonal faces, only the eight triangular truncation faces remain exposed. Thus the resulting surface has the following facets: 1*8 equilateral junction triangles and 3*16 arm triangles = 56 total.

The resulting {3,7} tilings are shown in Figure 2. All examples use a minimal amount of twist in each arm. Additional locally regular maps that maintain the same overall symmetry, and thus minimize the number of geometrically different tile shapes, can be obtained by giving all arms simultaneously the same amount of twist. For the tetrahedral frame construction depicted in Figure 2a, the arm twist would have to be augmented clockwise by a one-quarter Dehn twist to obtain the globally regular map that exhibits the full combinatorial symmetry of Klein's quartic curve.

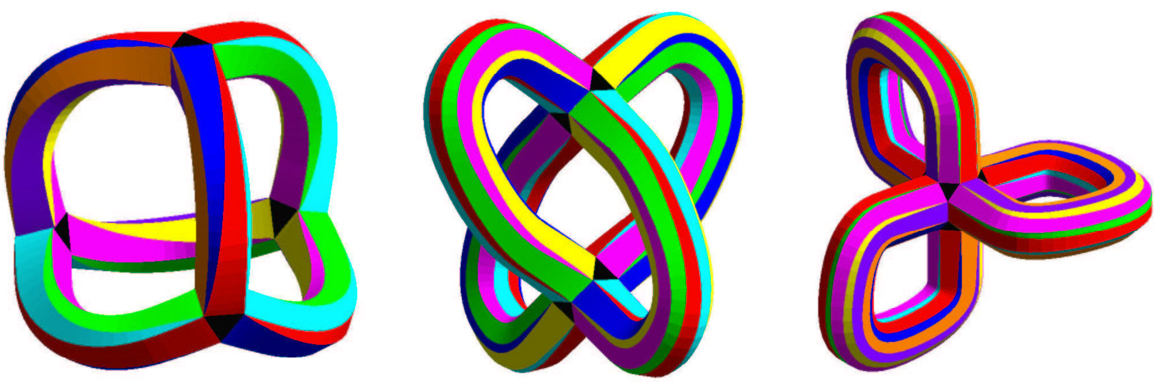

Figure 2: *Tubular constructions of genus-3 surfaces decorated with locally regular {3,7} tilings: (a) orientation-preserving tetrahedral symmetry, (b)* D_2 *symmetry, and (c)* D_3 *symmetry.*

As mentioned above, on a genus-3 surface there exist globally regular maps with 168 automorphisms (without reflections) for the {3,7} and its dual {7,3} tiling. One feature of a globally regular tiling is that all its Petrie polygons have the same length. Petrie polygons are closed zig-zag paths on the edges of the tiling that alternately make sharp left and right turns at subsequent vertices visited. Thus a Petrie path hugs any tile for exactly two consecutive edge segments. In a fully regular map, all of these Petrie polygons return to the starting point after the same number of moves, regardless at what vertex and in what direction one starts. For the Klein group on the genus-3 surface, the length of these Petrie polygons is always 8 – hence the name of Ferguson's sculpture: "The Eight-fold Way" [4].

This particular map is special in yet another way: it has the highest possible number of automorphisms of any map on a genus-3 surface. For genus 2 and higher, no map on a surface of genus *g* can have more than 84(*g*-1) orientation-preserving automorphisms. This theoretical limit established by Hurwitz [6] cannot be achieved for all values of *g*. The first such value for which it can be achieved is *g*=3; the next one is *g*=7; and the next one after that is *g*=14. Thus I am particularly motivated to find a nice symmetrical embedding of the corresponding regular map with 84*(7–1)=504 automorphisms on a genus 7 surface. However, since that task is rather more difficult than I had expected, we will gradually work our way up to higher-genus surfaces and focus first on artistically pleasing, locally regular maps.

3. General Junction Elements for {3,7} and {7,3} Tilings

Once we have a {3,7} tiling on a genus 3 surface, we can readily obtain its dual {7,3} tiling by picking a point in the middle of each tile and drawing a new edge across each original edge to each adjacent tile. This leads to a new set of modular components for the construction of many different {7,3} tilings. Such junction elements forming 3-way, 4-way, and 6-way junctions are shown in Figure 3. These flexible junction elements are covered with 6, 12, and 24 heptagon tiles, respectively. They all have half-arm stubs with serrate (zig-zag) endings with 4, 6, and 8 angled "teeth," respectively. Two such junction elements of the same kind can thus be joined in 4, 6, or 8 different ways, respectively. As we will see in the following, these junction elements can be used to form any type of surfaces of genus 2 or higher. Thus we next explore what other such general junction elements might exist.

Figure 3: *Different modular junction elements decorated with heptagons joining in valence-3 vertices: (a) 3-way junction, (b) tetrahedral junction, and (c) cube-based 6-way junction.*

The analysis is more easily done with the {3,7} constructions of Figure 2. All the arms must have even-valued rotational symmetries so that they can produce the valence-7 vertices required for the {3,7} tiling: Consecutive edges at the ends of the tubular arms alternately touch another arm or a (black) triangle of the junction body. By using 10-sided arms, we obtain a 12-way junction with a truncated dodecahedron as its junction body. By adding 6 loops to this junction, we then readily obtain a surface of genus 6.

This exhausts the possibilities of fully symmetrical junctions in which all arms have the same number of sides. It is possible to form somewhat less symmetrical junctions based on prismatic hubs: An *n*-sided prismatic hub would have *n* 4-sided arms plus two *n*-sided arms emerging from the polar faces.

4. Maps of Genus 4, 5, and 6

Using the construction techniques outlined in the previous section, one can construct "tubular" surfaces of any desired genus. By giving these tubular frames the overall structure of the edges on a regular prism or on an Archimedean or Platonic polyhedron, one can readily obtain triangle meshes of relatively high overall Euclidean symmetry. As one example, we look at the genus-5 case. A cube frame offers the highest symmetry for a tubular framework of this genus. It can be realized with eight 3-way junctions and twelve 4-sided anti-prism arms (Fig.4a). The total number of triangles is thus 8*2 + 12*8 = 112, and there will be a total of 48 valence-7 vertices. This forms a locally regular map. Each triangle can now be subdivided into three butterfly tiles, as Escher has done in one of his many tiling patterns, and as Dunham has done on the genus-3 surface [3]. Thus there will be a total of 336 butterflies on a genus-5 surface with the orientation-preserving symmetries of a cube (Fig.4b).

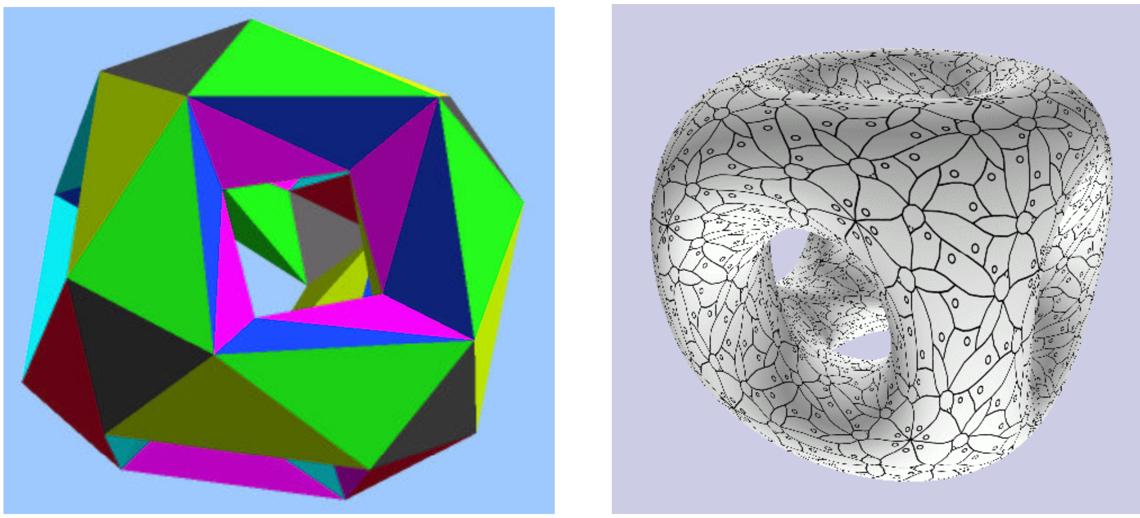

Figure 4: *Genus-5 surfaces with the rotational symmetries of a cube with locally regular {3,7} tilings: (a) the basic polyhedral construction, (b) an embellished version with Escher tiling.*

5. A Globally Regular Map on a Genus-4 Surface

To gain some experience with globally regular tilings, we study the regular map of 24 pentagons on a genus 4 surface with the symmetries of the group S_5, which is the permutation group of five objects. This topologically interesting object was pointed out to me by David Richter. On his website [11], he gives some beautiful illustrations of this object as a hyperbolic tiling on the Poincaré disc (Fig.5a), as well as an immersed self-intersecting star polyhedron called the dodecadodecahedron, which he implemented as a beautiful cardboard model (Fig.5b). This generalized polyhedron consists of an intertwined double shell comprising 12 large, mutually intersecting regular pentagons and 12 smaller star-pentagrams. Six sets of two pairs of these 5-sided faces are parallel to one another and are correspondingly colored with six different colors. Those same colors are also shown in the tiling of the Poincaré disc (Fig.5a). Figure 5c shows the same polyhedron, but makes the edges more visible.

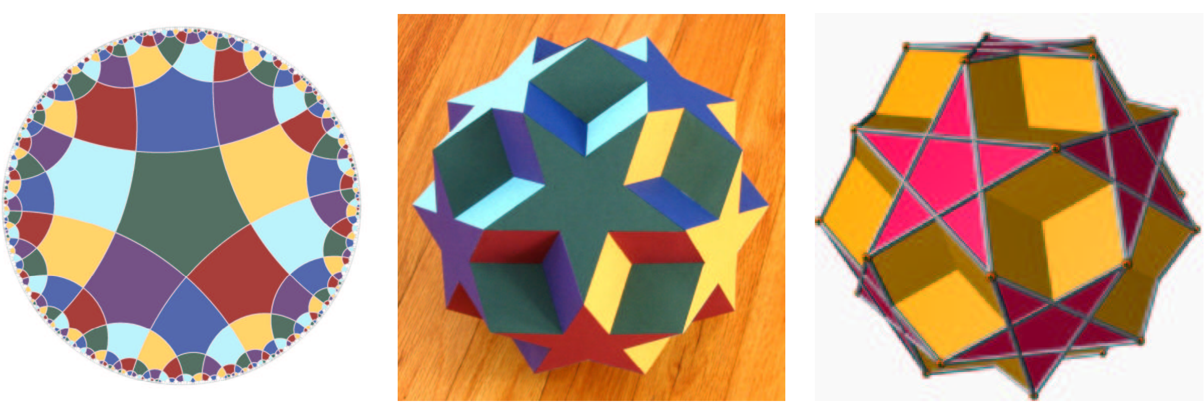

Figure 5: *A completely regular tiling of type {5,4} is possible on a self-intersecting genus-4 surface: (a) the basic pattern on the Poincaré disc, (b, c) on a self-intersecting dodecadodecahedron.*

Eulers formula: E–V–F=2*(genus-1), tells us that this polyhedron is of genus 4. With the goal of finding a true embedding of this tiling in 3D space, we switch to its dual structure: an orientable regular map of 30 quadrilateral tiles joining in 24 valence-5 vertices. Figure 6 shows one of several possible embeddings of locally regular maps with the desired characteristics. The displayed embedding exhibits symmetry D_{4h} and is based on a disk with 4 symmetrically placed holes and spokes. Among the other embeddings studied, one exhibit D_{5h} and is derived from the Lawson surface [5] of genus 4, – the configuration that would yield minimal overall bending energy, if the surface were realized as an embedded smooth 2-manifold. A third one has again symmetry D_{4h} and is conceptually equivalent to a prism with 4 longitudinal handles attached. However, the edges of none of these three meshes have the desired global flag symmetry that would make every edge equivalent to every other one, and which would thus exhibit all 120 orientation-preserving symmetries corresponding to the elements of the group S_5.

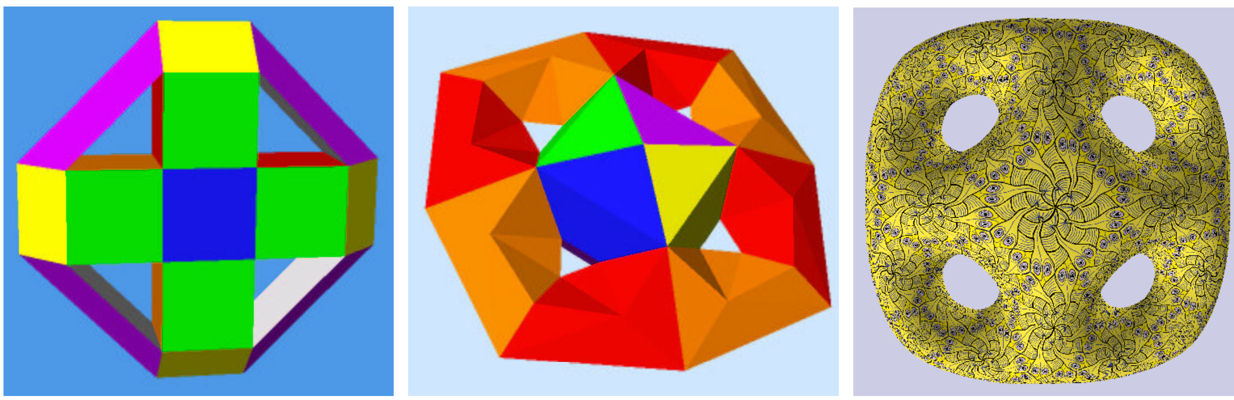

Figure 6: *Locally regular maps {4,5} and {5,4} on a surface of genus 4 with D_{4h} symmetry: (a) {4,5} on disk with 4 holes, (b) its polyhedral dual {5,4}, (c) corresponding Escher tiling.*

Because of the modular construction of these 2-manifold structures, one can now cut the various arms and apply fractional Dehn-twists to them to change the global connectivity of the network of edges, while maintaining local regularity in the tiling. One might suspect that not all combinations of different Dehn twists on the various arms need to be explored, and that some overall symmetry of type D_4 or D_5, respectively, should be maintainable in the embedded meshes that exhibit the desired high combinatorial symmetry. Under this assumption, there is only one Dehn-twist parameter for each structure that needs to be varied, and the number of cases to be analyzed is thus reasonably small.

Unfortunately, just twisting the arms as described above is not sufficient to find all possible symmetrical connectivities of these locally regular maps. One should also consider other possible Dehn twists that cyclically permute the connectivity of the mesh edges that run through the tunnels in our manifolds. This yields a much larger number of meshes that have to be examined whether they have the desired connectivity corresponding to the S_5 group. Analyzing these often very highly twisted meshes with sketches or by visual inspection of computer-generated edge-graphs is too tedious and error-prone to find the desired solution; and there is no guarantee that the desired solution is even among the maps that can be generated with the select fractional Dehn twists described above. I abandoned this approach after several hours of unsuccessfully staring the twisted meshes on the three embedded 2-manifolds mentioned above. However, we know that a solution must exist; and from Figures 5a and 5b, we even know the exact connectivity of the map as well as the length of its Petrie polygons (=6). Inspired by the high symmetry of the dodecadodecahedron (Fig.5b), I was then trying to find other highly symmetrical ways of embedding a genus-4 surface in 3D space – in particular one that would exhibit some 4-fold symmetry axes. In this context the cuboctahedron is a plausible starting shape. Conceptually, we can drill 4 separate tunnels through this Archimedean solid, connecting the four pairs of opposite triangle faces. Of course, if maximal symmetry is being maintained, then these four tunnels would all intersect at the center of the structure. Thus we need to bend these tunnels suitably to make them pass by one another without interference. This breaks the overall symmetry, but in a less objectionable way, since on the outside we can maintain the symmetry of the cuboctahedron. I actually started with a truncated octahedron, removed the 8 hexagonal faces, and replaced them with four twisted 6-sided prismatic tunnels made from 6 warped and stretched quadrilateral faces each. This yielded a mesh with 30 quadrilateral faces that all join in vertices of valence 5; thus forming yet another tiling of the required local connectivity (Fig.7a). But yet again, I was unable to convert this mesh into one that had the desired global connectivity by applying identical fractional Dehn twists to the quadrilaterals in the tunnel. An immediate indication of failure is when one finds a Petrie polygon that does not close on itself after 6 moves.

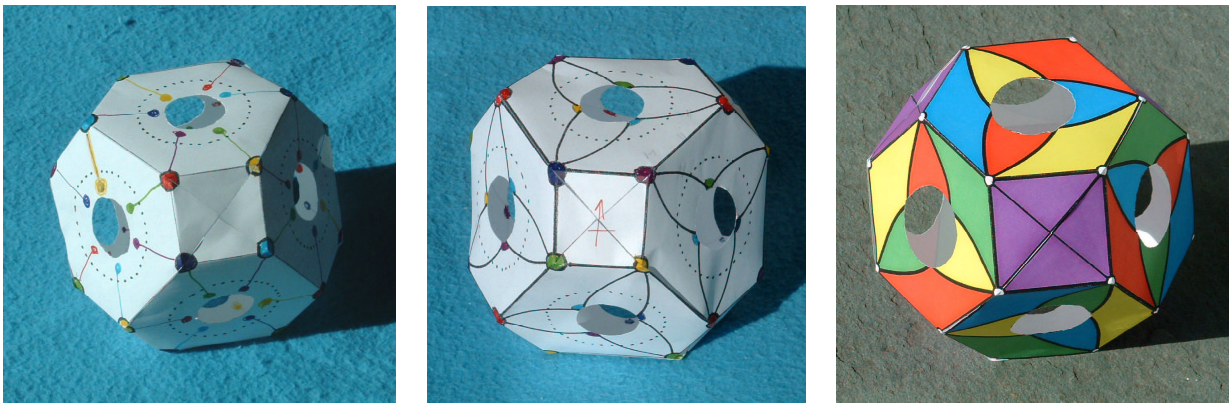

Figure 7: *Different {4,5} tilings on a genus-4 surface with octahedral symmetry: (a) locally regular map, (b) globally regular map, (c) same with colored faces.*

Many such failures inspired me to start the embedding process by drawing some of the desired Petrie polygons onto the selected embedding surface. A promising way was to make a zig-zag path of length 6 in each of the four tunnels (Fig.7b), as one would find on an anti-prism, rather than routing six "parallel" lines through these tunnels as in Figure 7a. I also made sure to place the endpoints of these anti-prisms so that the full orientation-preserving symmetry of the octahedron was maintained, and started to transfer the color pattern from Figure 5a onto the emerging meshing. Luckily, the edges of the resulting mesh joined up properly without contradictions! Finally with the mesh completed in a symmetrical manner, I was able to trace out several different Petrie polygons and determine their lengths. All of them were of length 6 !

The colored vertices in Figures 7a and 7b corresponds to the face colors of the dodecadodecahedron shown in Figure 5a and 5b, and they exhibit a symmetrical distribution over the truncated octahedron. Note that all the Petrie paths in Figure 7b pass through vertices comprising all six different colors. It is natural to ask whether there exists an equally symmetrical coloring for the edges and/or the faces of this map. Indeed I have found a face coloring using 5 colors, for which all 5 colors are present at every vertex, and for which every face is surrounded by the four other colors (Fig.7c).

6. Genus-7 Maps

Fortified with the experience of finding a highly symmetrical embedding of the globally regular {4,5} tiling on a genus-4 surface, we now attempt to find the Hurwitz map with 504 automorphisms on a surface of genus 7. The first step was to construct a hyperbolic {3,7} tiling on the Poincaré disc and try to produce a map with the desired combinatorial symmetries. From sources such as [1] and [9] it was clear that such a map would have 72 unique vertices and a Petrie length of 18. Finding a corresponding assignment of 72 different identifiers to the vertices of the hyperbolic tiling was a little tedious, but manageable. Using a good repetitive coloring pattern with 8 colors, was very helpful in this task. Transferring this map onto an embedded surface of high Euclidean symmetry, while respecting that symmetry as much as possible, is the real challenge.

We start with the construction of several surfaces with high symmetry and then try to decorate them with locally regular {3,7} tilings that respect those geometric symmetries. Examples of tubular frameworks of genus 7 are shown in Figure 8. These surfaces are built in a manner similar to the genus-3 surfaces depicted in Figure 2, using junction elements with 3 and 4 arm stubs, respectively. Of course, the high genus of value 7 allows us now to also use junctions of higher valence – up to a valence of 14, where we would have only a single junction with seven loops attached, similar in spirit to the surface depicted in Figure 2c. Table 1 lists the conceptual possibilities for constructing a tubular framework of genus 7 from junctions that all have the same branching valence.

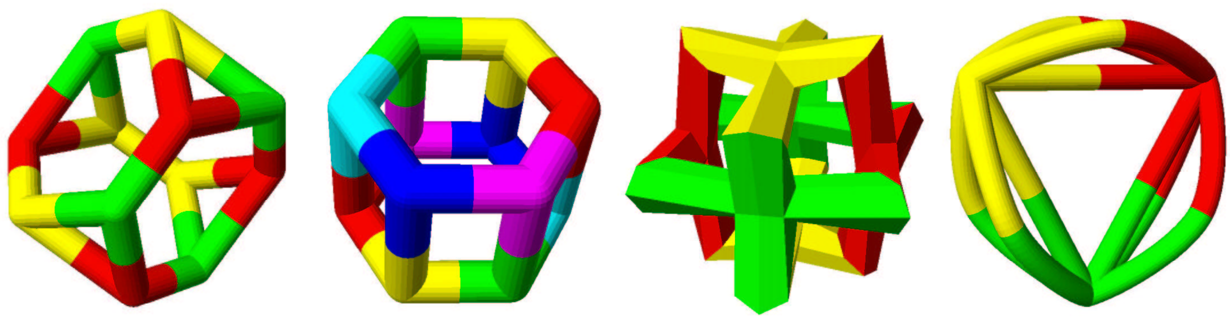

Figure 8: *Composition of junction elements from Figure 2 into symmetrical genus-7 surfaces: (a) truncated tetrahedron, (b) hexagonal prism, (c) octa-frame, (d)ring of cuboid junctions.*

Table 1 first lists the valences and number of the tubular junctions needed to obtain a structure of genus 7. We then assume that these junctions are formed as semi-regular polyhedra with regular *s*-gons to which the *s*-sided anti-prism arms connect. The table also lists the number of tubular arms and their rotational symmetry # (*s*). Since the overall number of triangles must always be 168, we can calculate how many extra triangles need to be supplied by the junction polyhedron (column 4).

Option	Junction valence	Junction count	Junction triangles	Arm prism #	Arm count	Arm triangles	Overall triangles
A -3prism	3	12	24	4	18	144	168
B -tetra	4	6	24	6	12	144	168
C	5	4	28	7	10	140	168
D -cube	6	3	24	8	9	144	168
E -octa	8	2	24	9	8	144	168
F	14	1	0	12	7	168	168

Table 1: *Potential numerical characteristics of the locally regular {3,7} maps on a genus-7 surface.*

Options A, B, and D use the same kind of junction building blocks discussed in Section 2 – but employ three times as many of these junctions as for the genus-3 case:

- **Option A** can be realized as a frame in the shape of a truncated tetrahedron (Fig.8a) or of a hexagonal prism (Fig.8b) and thereby preserve a high degree of symmetry in the embedding surface.
- **Option B** is topologically an octahedral tube frame (but has only half as many symmetry elements); it looks like the most plausible choice for making a highly symmetrical genus-7 surface (Fig.8c).
- **Option D** with only 3 junctions implies that pairs of junctions must be connected with 3 arms each. This does not achieve very high symmetry overall (Fig.8d).

While cases C, E, F also seem numerically possible, they appear not to be realizable as a locally regular tiling structure that covers all the arms and all the junctions in the same way:

- **Options C and E** have arms with an odd symmetry count, This means that the valence-7 vertices do not emerge naturally as in the preceding constructions: The ends of the arms can no longer have every second edge directly connected to a neighbor arm and the edges in between to a junction triangle.

- **Option F** causes difficulties with the realization of a symmetrical hub: How does one connect the 14 ends of seven arm loops in a single junction? A cuboctahedral hub would have the desired 14 facets, but the two types of faces have a different number of neighbors and would lead to different arms.

Since junction valence and junction count can only change in integer steps, Table 1 exhausts all options.

When studying the meshes resulting from Options A, B, and D, none of them immediately exhibited the desired global connectivity of the Hurwitz group of order 504. As discussed for the genus-3 case, I then tried to twist the arms in angular increments given by the sidedness of the anti-prism arms. Alas, I could not find any twist value that would make all possible Petrie polygons of the same length. Again, visual inspection was tedious and even more error prone, and there is no guarantee that a tiling exists that simultaneously has the desired global symmetry properties and also reflects the Euclidean symmetries of the chosen embedding surface.

It seems most promising that such a solution may exist, if the embedding surface has symmetry axes of valence 3 or 7. Since the investigation of the mapping of the group S_5 onto a genus-4 surface indicated that a set of well chosen tunnels through a sphere-like starting shape can lead to the most symmetrical embedding of a locally regular map, I am planning to apply a similar approach to the Hurwitz group of order 504. Currently I am most hopeful to find a solution on a genus-7 manifold with 7-fold prism symmetry of type D_7, – perhaps with seven tunnels in the form of a star-heptagram, where the tunnels advance by *j* sectors of 360/7 degrees, thus accommodating the required end-around connections.

7. Discussion and Conclusions

The search for a nice symmetrical embedding of the Hurwitz group of order 504 on a genus-7 surface turned out to be much more difficult than I anticipated at the conclusion of the last Bridges conference in London, when I decided to make this the quest for this year's conference [13]. Even trying to find a globally regular solution for the smaller problem of 24 pentagons on a genus-4 surface was a challenging task. I am not aware of any automated approaches that can solve these kinds of problems, and a brute-force computer search has not yet been attempted.

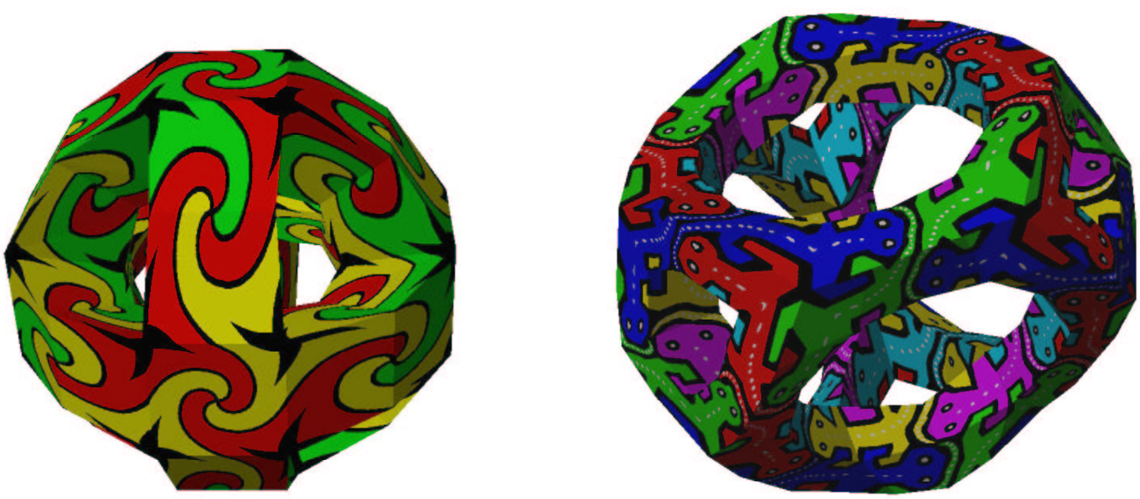

Figure 9: *Hyperbolic Escher tilings: (a) {5,4} starfish pattern on a genus-7 octahedral frame; (b) 72 Newts on a genus-7 surface with the structure of a truncated tetrahedron.*

Among the various means that I have used to find solutions to the many puzzles discussed in this paper, frequent switching between different media seemed to be most productive to yield new insights as well as pleasing, although not globally flag-symmetrical, solutions (Fig.9). The main media utilized were: sketches with colored pens, computer graphics renderings, and 3-dimensional paper models; basic patterns for the paper models were often designed as modular components on the computer. For those problems where finding a map with global flag symmetry was the main goal, the first step was always to get a representation on the Poincaré disc and determining the length of the Petrie polygons for the particular problem. It was further useful to obtain some regular coloring of the facets or vertices on the Poincaré disc, and finding other cyclic pattern loops that run from border to border of the fundamental domain; these represent cycles that cannot be collapsed and which thus must circumnavigate at least one arm or one hole of the 2-manifolds considered for embedding.

The quest continues to find a pleasing embeddable model of the Hurwitz group of order 504. Thanks to publications such as [1] and [9], I understand its symmetries and the length of its Petrie polygons. But it is still not clear what symmetry in Euclidean space the surface should have on which a corresponding tiling can be embedded while fully respecting the chosen 3D symmetry. The latest results will be presented at the conference. And beyond that, there are a few more challenges waiting – concerning globally regular tilings of genus 9 and 11 [14].

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