# Hyperseeing the Regular Hendecachoron

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## Abstract

The hendecachoron is an abstract 4-dimensional polytope composed of eleven cells in the form of hemi-icosahedra. This paper tries to foster an understanding of this intriguing object of high symmetry by discussing its construction in bottom-up and top down ways and providing visualization by computer graphics models.

## 1. Introduction

The hendecachoron (11-cell) is a regular self-dual 4-dimensional polytope [3] composed from eleven non-orientable, self-intersecting hemi-icosahedra. This intriguing object of high combinatorial symmetry was discovered in 1976 by Branko Grünbaum [3] and later rediscovered and analyzed from a group theoretic point of view by H.S.M. Coxeter [2]. Freeman Dyson, the renowned physicist, was also much intrigued by this shape and remarked in an essay: “Plato would have been delighted to know about it.”

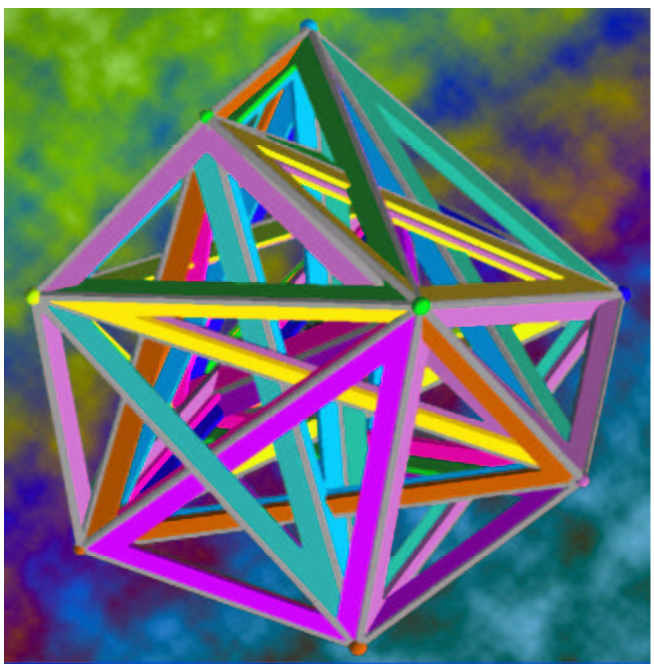

**Figure 1**: Symmetrical arrangement of five hemi-icosahedra, forming a partial Hendecachoron.

For most people it is hopeless to try to understand this highly self-intersecting, single-sided polytope as an integral object. In the 1990s Nat Friedman introduced the term *hyper-seeing* for gaining a deeper understanding of an object by viewing it from many different angles. Thus to explain the convoluted geometry of the 11-cell, we start by looking at a step-wise, bottom-up construction (Fig.1) of this 4-dimensional polytope from its eleven boundary cells in the form of hemi-icosahedra. But even the structure of these single-sided, non-orientable boundary cells takes some effort to understand; so we start by looking at one of the simplest abstract polytopes of this type: the hemicube.

## 2. The Hemicube

The hemicube is constructed by starting with only half a cube (Fig.2a), in particular the three faces clustered around one of its vertices (D), and gluing the six open edges of this 2-manifold onto each other, so that opposite edges and vertices fall onto one another, as indicated in Figure 2a with the three pairs of arrows with matching colors. Thus we obtain an object with 4 vertices, 6 edges, and 3 quadrilateral faces. The skeleton of this abstract polytope, composed of the vertices and edges of this object, corresponds to the graph $K_4$, the complete graph with four nodes, in which every node is connected to every other node (Fig.2b). In 3 dimensions this can be represented as a tetrahedral frame, i.e. the skeleton of the 3-dimensional simplex (Fig.2c). If we label the 4 vertices as A, B, C, and D, then the three quadrilaterals have the contours: ABCD, ABDC, and ADBC. Clearly these quadrilaterals are not planar, but can be realized as bilinear Coons patches, and they will all intersect with one another. Topologically, this object has the same connectivity as the non-orientable projective plane (on which you can walk North, pass through infinity, and return from the South – but upside down).

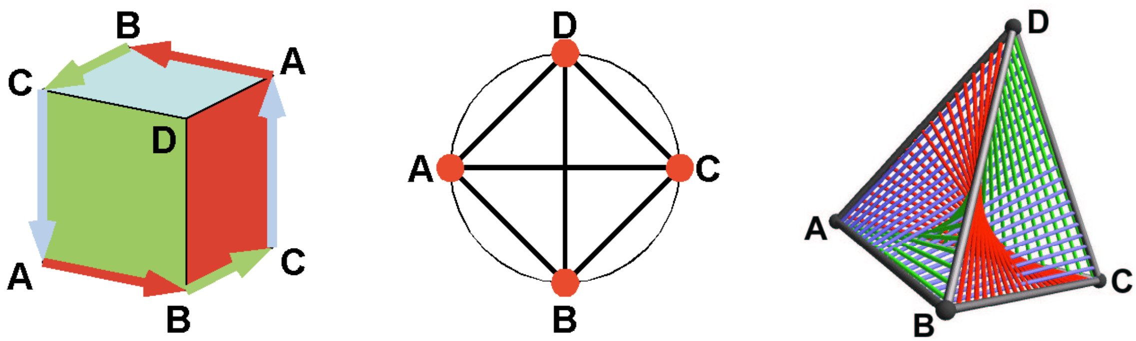


**Figure 2**: (a) Hemicube; (b) complete graph $K_4$; (c) tetrahedral frame with three saddle faces.

## 3. The Hemi-icosahedron

In a way analogous to the above, we can construct an abstract polytope from half an icosahedron (Fig.3a). Again we glue the open edges along the boundary onto each other, so that opposite edges and vertices are being joined. This yields a structure with 6 vertices, 15 edges, and 10 triangular faces. The skeleton of this abstract polytope forms the complete graph $K_6$ (Fig.3b). This corresponds to the 5-dimensional simplex. It can be projected into 3-dimensional space as an octahedron that has been slightly warped (Fig.3c), so as to avoid intersections of the 3 edges going through its inner space [7]. If the 6 vertices are labeled: A, B, C, D, E, and F, then the 10 faces will be the triangles: ABC, ACE, ADF, AED, AFB, BDC, BED, BFE, CDF, and CFE; these are half of all the faces that would appear in the complete 5-dimensional simplex. The other ten faces also form a proper hemi-icosahedron!

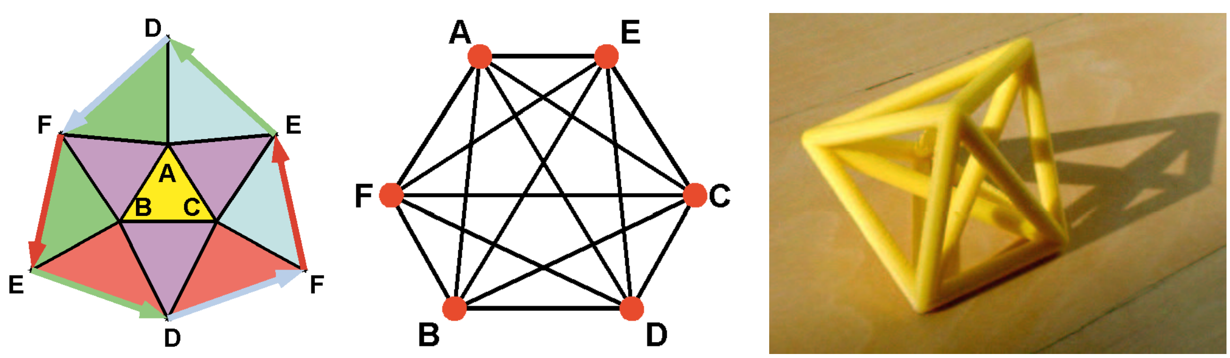


**Figure 3**: (a) Hemi-icosahedron; (b) complete graph $K_6$; (c) asymmetrical octahedral frame.

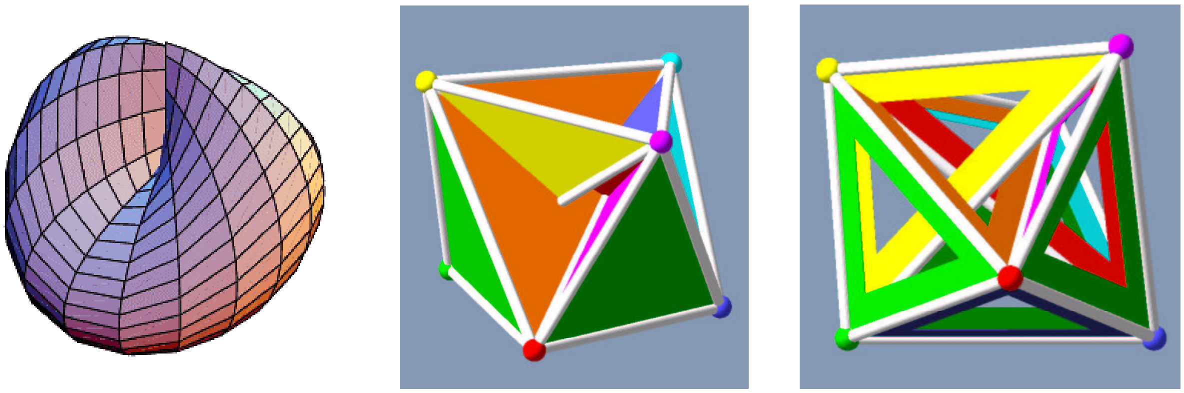

**Figure 4**: (a) Crosscap; (b,c) two views of the hemi-icosahedron projected into 3D space.

In an attempt to visualize the hemi-icosahedron as much as an actual "surface" as possible, one can map it onto one of the classical representations of the projective plane with the same topology – the crosscap (Fig.4a). The 2-manifold composed of 10 triangles shown in Figures 4b and 4c similarly has a "hemispherical bottom" with some structure on top of it that lets the surface cross through itself, so that it becomes single-sided – a so called *Whitney Umbrella* [9]. In Figure 4c, the middle part of all triangles has been cut out, so that only a narrow frame is left. This let's us see inside this self-intersecting polyhedron.

But there is a more symmetrical representation of the projective plane which fits the connectivity of the hemi-icosahedron even better: Steiner's Roman surface (Fig.5a). This representation retains four tetrahedral faces of an octahedron and adds to those the three medial squares that connect the triangles into a non-orientable 2-manifold. These quadrilaterals are partitioned into two triangles each to yield a total of ten triangles. Such a hemi-icosahedron is shown in Figure 5b – again with cut-outs in the triangular faces; but here we look into one of the concavities of the Roman surface, rather than onto a tetrahedral face as in Figure 5a.

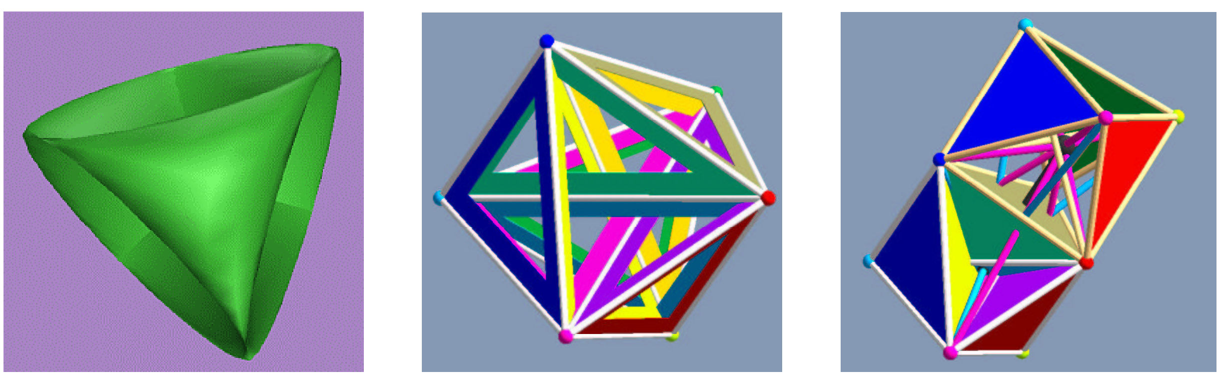

**Figure 5**: (a) Steiner surface; (b) a "Steiner" hemi-icosahedron; (c) two such cells glued together.

## 4. Bottom-up Construction of the Hendecachoron

To assemble a complete hendecachoron, one can start with an initial hemi-icosahedron and glue a copy onto each one of its 10 triangular faces. In 5-dimensional space the hemi-icosahedron, just as the skeleton of the simplex, is totally regular; all edges are of equal length, and all faces are equilateral. Thus two of these cells can readily be glued together by joining two triangular faces. Now we add a third hemi-icosahedral cell to this assembly so that it also shares one of the edges that the first two polytopes already share. Since the dihedral angle of the hemi-icosahedron is 70.53° (the same as the tetrahedron), there is a large wedge of empty space left along that edge between the third cell and the first cell in this assembly. If we now try to forcefully close this wedge of free space, then the assembly of three hemi-icosahedra will have to bend into the next higher dimension, where the three polytopes form a 3-fold symmetric constellation around the shared edge. This process must now be repeated on all the edges of the hemi-icosahedron.

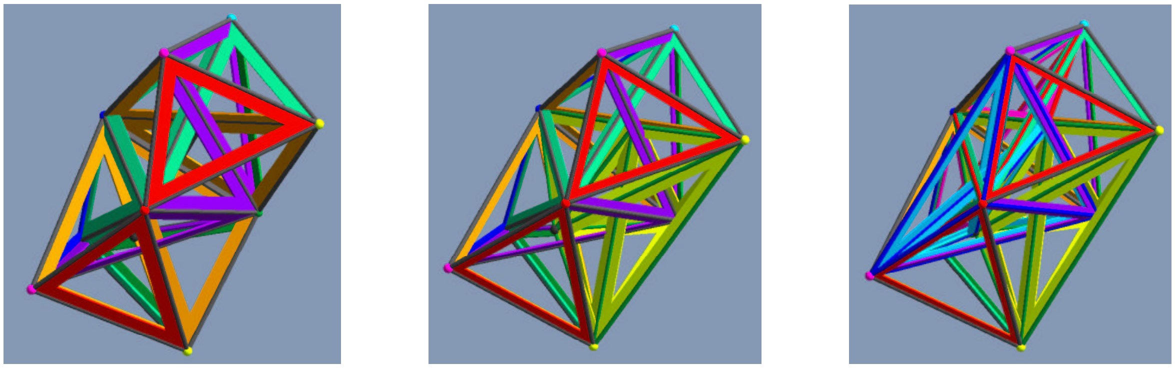

**Figure 6**: (a) The two cells of Figure 5 with cut-out faces; (b) a third cell added; (c) five cells total.

The assembly of eleven hemi-icosahedra is thus warped into a higher-dimensional space so that all the free-space wedges along all the edges can be closed, and every cell shares a triangle face with every one of the other 10 cells. The enforced bending is so strong that the whole assembly curls up through itself, with opposite faces falling onto one another, so that in the end there is no face left unmatched, and every edge in this structure has exactly 3 hemi-icosahedra around it. The resulting abstract polytope is the regular hendecachoron. It has 11 vertices, 55 edges, 55 triangular faces, and 11 hemi-icosahedral cells; considered as a 4-dimensional object, it is thus self-dual. Of course all those cells wildly penetrate one another, and one would have to go to 10-dimensional space to exhibit this object without any self-intersections. In 3D space it is difficult to make sense of this cluster of mutually intersecting edges and faces. However, if this assembly is shown step by step, one hemi-icosahedron at a time, then one can get an idea how all of this fits together.

Figure 5 shows the start of such a process. In Figure 5c a second hemi-icosahedron has been added onto the tan surface of a first Steiner cell. In the latter cell only the outer tetrahedral faces are shown; the inner 6 triangles have been omitted to reveal all the edges. This figure also shows the two additional vertices necessary to construct the complete hendecachoron; these vertices have been placed at the centers of the two Steiner cells. To continue the build-up of a complete 11-cell, we next add three more hemi-icosahedra on the three "sides" of this twin anti-prism structure. These cells re-use two of the outer faces as well as four of the outer vertices plus the two inner centroid vertices (Fig.6b). The assembly of the two original cells and the three new ones is shown in Figure 6c. Finally we need 6 more cells; they each need to use one each of the six inner triangles of the two Steiner cells and various faces from the three outer cells added in the previous step. But the structure gets rather cluttered after the first five cells, and it is not clear whether showing all the triangles adds to the understanding of the hendecachoron.

Another approach to assemble a 11-cell is to start with a central Steiner cell (Fig.7a) and, in a first phase add four additional cells onto the 4 tetrahedral faces of this cell. However, since we already have used six vertices to start with, and can add only 5 more, these 4 additional cells will have to share many vertices. The most symmetrical way to add 5 more vertices to the original set of 6 is to add a (white) vertex above each tetrahedral face, and one more (black) in the center of the first Steiner cell (Fig.7c). The four additional cells thus need to penetrate the original one to reach the three vertices on the "other" side from the face that they share with the original cell. Figures 7b and 7c show the state after a first such cell has been added to the original one; Figure 7c also shows the additional edges that will be used by the other three cells added in this second phase. The result can be seen in our frontispiece (Fig.1). Finally, we need to add six more hemi-icosahedra, and these are exactly those six cells that share the (black) vertex in the center of the structure. But again, the structure gets so cluttered that a small-scale rendering does not make a useful contribution to the understanding of the 11-cell.

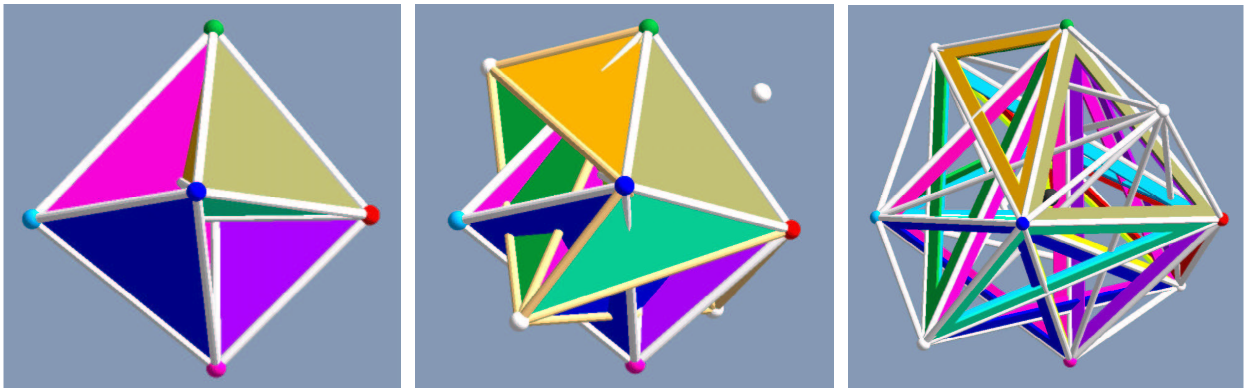

**Figure 7**: (a) One Steiner hemi-icosahedron; (b) a second penetrating cell added on the tan triangle; (c) a view with cut-out faces to reveal the inside and the new edges needed for the next three cells.

## 5. Top-down Construction of the Hendecachoron

In the spirit of *hyperseeing*, we present yet another approach to gain some more insight into this intriguing abstract polytope. Consider the skeleton of edges and vertices only. Since there are 11 vertices and 55 edges, this represents the complete graph, $K_{11}$, and this graph can be seen as one possible projection of the skeleton of the 11-Cell – or, alternatively also as the skeleton of the 10-dimensional simplex. However, the 10D simplex would use all {11 choose 3}=165 triangles; while the 11-cell only uses 1/3 of those faces. Thus the hendecachoron represents a subset of all the elements present in the 10D simplex (or its projections into lower-dimensional spaces).

At every vertex there are thus present: 10 edges, 15 faces, and 6 hemi-icosahedral cells; and the vertex figure, i.e. the intersection shape that we would get if we surrounded one of these vertices with a small sphere, would be an abstract polytope with 10 vertices, 15 edges, and 6 pentagonal faces. This polytope does indeed exist and is known as the hemi-dodecahedron [1],[3] – it is the dual of the hemi-icosahedron and can be constructed in an exactly analogous manner (see Section 7) as the hemicube or the hemi-icosahedron (see Sections 2 and 3).

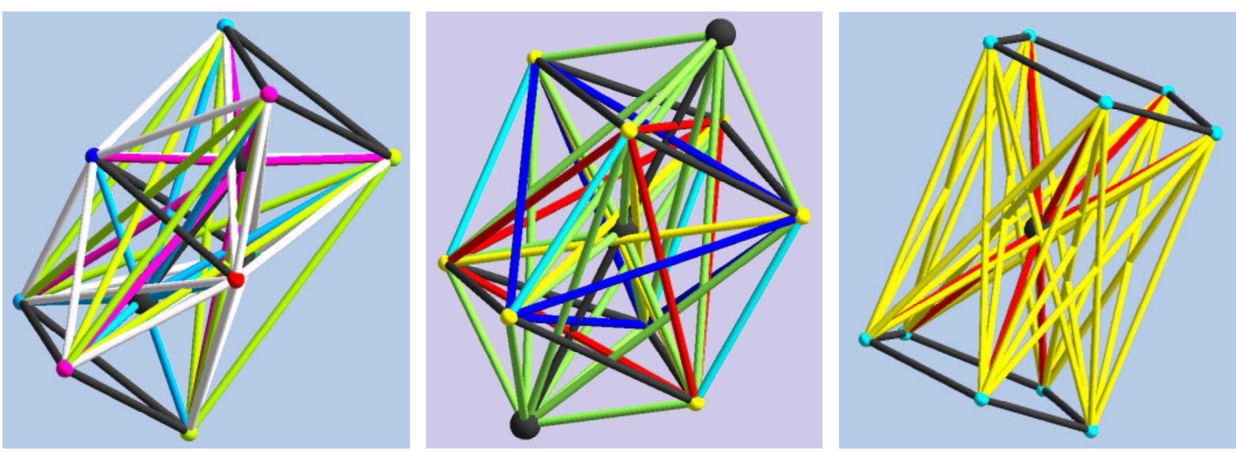

**Figure 8**: Prismatic arrangements of 11 points in 3D with (a) 3-fold, (b) 4-fold, (c) 5-fold symmetry – enhanced with the edges of the complete graph $K_{11}$; some asymmetry to avoid edge coincidences.

The above discussion completely defines the connectivity of the hendecachoron. But the real challenge still is to find a way to make a 3-dimensional visualization model of the whole thing. Clearly, the placement of the eleven vertices in 3D space is a crucial choice for a good visualization. One would like to preserve as much symmetry as possible, yet at the same time avoid too many coincidences that mask some edges or vertices. Putting all eleven vertices on a circle in a plane, makes all 55 edges easily visible, but yields a pretty useless visualization of the hendecachoron, since its triangular facets all lie in the same plane. The key is to find a 3-dimensional arrangement of the eleven points that yields a low variation on the individual edge lengths. The arrangements shown in Figure 8 come readily to mind: They all have prismatic symmetry (rotational symmetry around a dominant axis); they have 3-, 4-, and 5-fold symmetry, respectively, with 2, 3, and 1 of the vertices lying on the rotation axis.

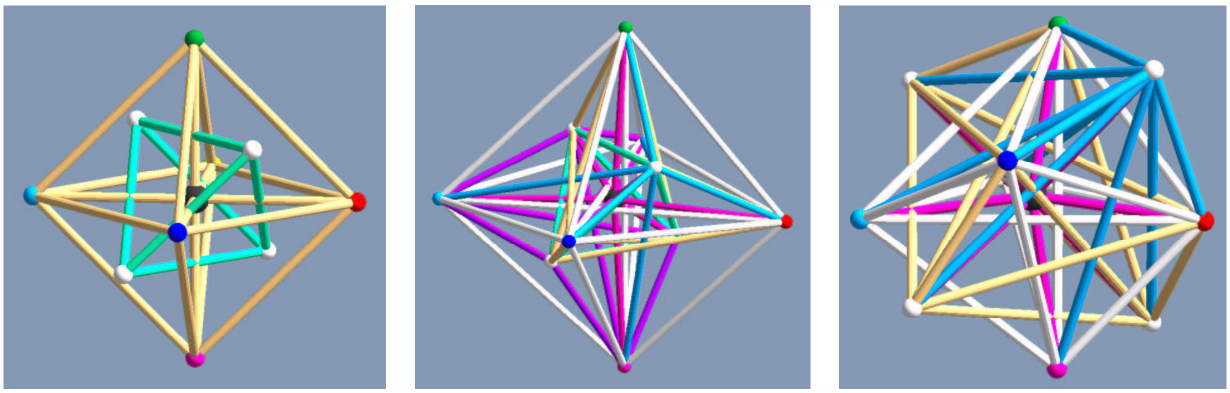

**Figure 9**: Spherically symmetrical arrangement of eleven points in 3D space; (a) the Plato shells, (b) the complete edge graph among all eleven nodes, (c) ten of the nodes lie on a common sphere.

We have also tried to find point arrangements with a more "spherical" symmetry, i.e. based on the symmetries of the Platonic solids. The only good solution found has the points located on three concentric shells: six points in an octahedral configuration, four points in a tetrahedral arrangement, and the last point at the center of this assembly (Fig.9a). In Figure 9b we have added all the edges of the complete graph $K_{11}$ among the eleven vertices, and a slight asymmetry has been added into the octahedral frame to prevent the edges between opposite poles to pass through the center vertex. In Figure 9c the four vertices of the tetrahedral frame have been moved outwards to fall also onto the circum-sphere of the octahedron. This is the arrangement underlying the structure shown in Figure 7b. In this framework of points and edges we can now display all 55 faces as narrow triangular frames. The ten faces belonging to the same hemi-icosahedron have been given the same color (Fig.10). However, since each face is shared by two cells and the hemi-icosahedra are single-sided (non-orientable) surfaces, the resulting visible coloring is somewhat arbitrary, and each face would have two different colors on its two sides. Tom Ruen has recently added a corresponding coloring [6] to the hemi-icosahedron faces visible in Coxeter's diagram (Fig.1 in [2]) showing the connectivity of these 11 cells.

## 6. Ultimate Visualizations

The complete assembly of 11 hemi-icosahedra will always remain highly cluttered. It is not clear that reducing the faces to narrow triangular frames can provide enough visibility into the interior structure to yield a decent visualization model of the hendecachoron. This structure is so intricate that perhaps no static visualization model is good enough to represent it. Perhaps a movie has a better chance. One could show a step-by-step addition of individual hemi-icosahedral cells. Alternatively one could produce a sequence of cut-away views of the whole object; this could be done with the final 3D projection, or it could even be done in 4D space by passing a 3D cutting hyper-plane through the hendecachoron. This might give some deeper insights into its internal connectivity.

Probably even better than a pre-recorded movie would be an interactive computer graphics application that lets the users move such a cutting plane back and forth through the hendecachoron at their whim. The proprioceptic feedback between the hand controlling the cutting-plane position and the observation of the resulting cut-figure yields an even deeper understanding of the examined structure.

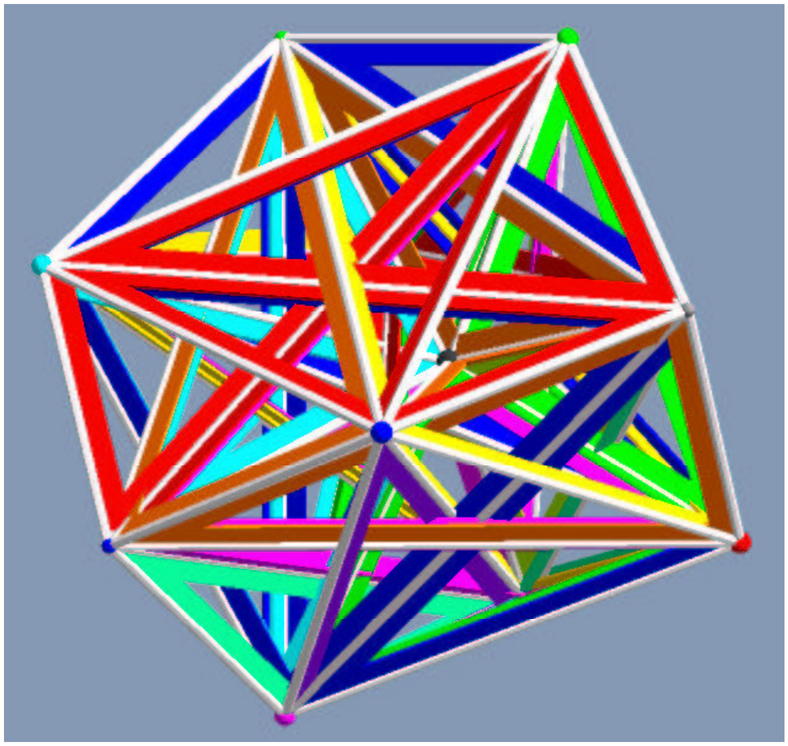

**Figure 10**: The full face set of the hendecachoron suspended in the skeleton of the 10D simplex.

## 7. The Cluster of 57 Hemi-dodecahedra

Real and abstract polytopes are often used as visualization aids to demonstrate the symmetries of a particular algebraic group. The hendecachoron has the combinatorial symmetries of the projective linear group $L_2(11)$ of order 660 [2]. Every cell of this polytope can be mapped onto any other one of its cells, and in any orientation permissible by the symmetries of the hemi-icosahedron itself. Since the latter one has 60 symmetries (10 triangles with 6 symmetries), the hendecachoron has a total of 660 automorphisms.

For completeness it should be pointed out that there is one other object of the same kind associated with the projective linear groups $L_2(q)$ where $q$ is a prime power. It is 57-Cell discovered by H.S.M. Coxeter [1], and it is also built from self-intersecting non-orientable cells in a combinatorially completely symmetrical manner. It corresponds to the case $q$=19. In Section 5 we encountered the hemi-dodecahedron as the vertex figure of the hendecachoron. This hemi-dodecahedron also is a regular abstract polytope; it has 10 vertices, 15 edges, and 6 pentagonal faces. It is constructed by starting with half of a regular dodecahedron (Fig. 11a) and again joining pair-wise opposite points on the open perimeter of this 2-manifold. Figure 11b illustrates how this might be done: take the outer-most vertices of the 5 pentagons attached to the central pentagon in Figure 11a and pull them across the center and join them with their opposite twin. This results in the connectivity of the Petersen graph, in which two such stretched pentagons join in an edge highlighted with an arrow in Figure 11b. Of course, this edge diagram should again be seen as a roughly spherical cluster that can act as a boundary cell for a higher-dimensional object (Fig.11c).

57 of these hemi-dodecahedral cells can be glued together to form the boundary of a "spherical" object in 4D space. Because in this cluster there are always five cells joining around a common edge, the wedge-shaped gap left open is narrower; thus less curvature in 4D space is required to close it, and it takes more cells to form the 4D "spherical cluster" that closes back onto itself. – It certainly is astounding that this 4D shell does indeed close perfectly onto itself without leaving any gaps or holes! The resulting object is Coxeter's 57-Cell with 57 vertices, 171 edges, 171 faces, and 57 hemi-dodecahedral cells [1].

Not too surprisingly, its vertex figure is the hemi-icosahedron; thus at each of its 57 vertices, there join together six edges, 15 pentagonal faces, and 10 hemi-dodecahedral cells. This object is another regular self-dual abstract polytope. In some way it is the big sibling of the regular hendecachoron. Recently Leemans and Schulte have proven [5] that these two polytopes are the only two that belong to the linear projective group $L_2(q)$ with $q$ a prime power, and that there can be no other ones of this type.

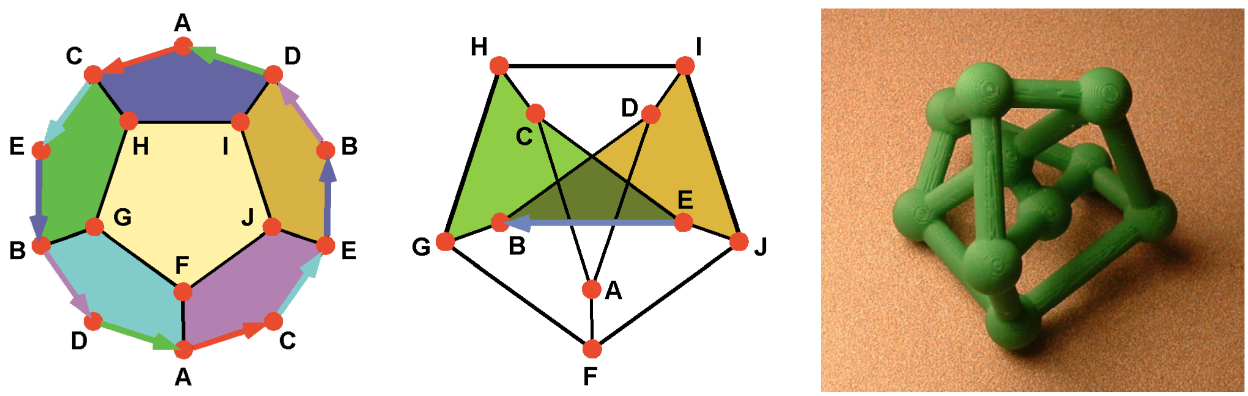


**Figure 11**: (a) Hemi-dodecahedron; (b) Petersen graph in the plane, (c) embedded in 3D space.

With respect to the hemi-dodecahedron shown in Figure 11, Lajos Szilassi has recently solved a puzzling question originally raised by Branko Grünbaum: Can the this abstract 6-sided polytope be realized as a non-degenerate embedding of a self-intersecting polyhedron in 3D space? Szilassi found a solution in which five of the six faces are simple planar (concave) polygons and only one face is self-intersecting (but still planar) [8].

## Acknowledgements

The authors would like to thank the many people who have taken the time to read and comment on our original draft of this paper after a short synopsis appeared in Jaron's World in Discover magazine [4] which mentioned this strange and hard-to-visualize "hendecatope". Based on several convincing arguments we have since switched to the more descriptive name – the "regular hendecachoron."

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