Adaptive Anisotropic Remeshing for Cloth Simulation: Supplementary Material

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1 Simplifying the augmented Lagrangian

The augmented Lagrangian is (see main text):

$$\mathcal{L}_A(\mathbf{x},\mathbf{s};\boldsymbol{\lambda},\mu) = f(\mathbf{x}) + \boldsymbol{\lambda}^T\hat{\mathbf{g}}(\mathbf{x},\mathbf{s}) + \frac{\mu}{2}\|\hat{\mathbf{g}}(\mathbf{x},\mathbf{s})\|^2, \quad (1)$$

where $\hat{\mathbf{g}}(\mathbf{x},\mathbf{s}) = \mathbf{g}(\mathbf{x}) + \mathbf{s}$. In terms of $\mathbf{s}$, $\mathcal{L}_A$ is simply a quadratic function over the domain $\mathbf{s} \geq 0$. In fact, all the components of $\mathbf{s}$ are independent: we can write

$$\mathcal{L}_A(\mathbf{x},\mathbf{s};\boldsymbol{\lambda},\mu) = f(\mathbf{x}) + \sum_{j=0}^{m}\left(\lambda_j(g_j(\mathbf{x})+s_j) + \frac{\mu}{2}(g_j(\mathbf{x})+s_j)^2\right). \quad (2)$$

Therefore, to minimize $\mathcal{L}_A$ with respect to $\mathbf{s}$, it suffices to find the minimizer of the function

$$L(s) = \lambda(g+s) + \frac{\mu}{2}(g+s)^2, \quad (3)$$

where g may be treated as a constant. As L is convex and $L'(s) = \lambda + \mu(g+s)$, the optimal $s \geq 0$ is simply

$$s^* = \max\left(-g - \frac{\lambda}{\mu}, 0\right). \quad (4)$$

When $g+\lambda/\mu \leq 0$, we have $s^* = -g-\lambda/\mu$, and the corresponding minimum value of L is

$$\begin{aligned} L(s^*) &= \lambda(g+s^*) + \frac{\mu}{2}(g+s^*)^2 && (5)\\ &= \lambda(g-g-\lambda/\mu) + \frac{\mu}{2}(g-g-\lambda/\mu)^2 && (6)\\ &= -\frac{\lambda^2}{2\mu}. && (7)\end{aligned}$$

On the other hand, if $g + \lambda/\mu \geq 0$, we find that $s^* = 0$, and so

$$\begin{aligned} L(s^*) &= \lambda(g+s_j^*) + \frac{\mu}{2}(g+s_j^*)^2 && (8)\\ &= \lambda g + \frac{\mu}{2}g^2 && (9)\\ &= \frac{\mu}{2}(g+\lambda/\mu)^2 - \frac{\lambda^2}{2\mu}. && (10)\end{aligned}$$

It can be verified that both cases are subsumed under

$$\begin{aligned} L(s^*) &= \frac{\mu}{2}\big(\max(g+\lambda/\mu,0)\big)^2 - \frac{\lambda^2}{2\mu} && (11)\\ &= \frac{\mu}{2}\tilde{g}^2 - \frac{\lambda^2}{2\mu}, && (12)\end{aligned}$$

where we define $\tilde{g} = \max(g+\lambda/\mu, 0)$. Applying the same argument to the original augmented Lagrangian yields

$$\begin{aligned} \mathcal{L}_A(\mathbf{x};\boldsymbol{\lambda},\mu) &= \mathcal{L}_A(\mathbf{x},\mathbf{s}^*;\boldsymbol{\lambda},\mu) && (13)\\ &= f(\mathbf{x}) + \frac{\mu}{2}\|\tilde{\mathbf{g}}(\mathbf{x})\|^2 + \frac{\|\boldsymbol{\lambda}\|^2}{2\mu}, && (14)\end{aligned}$$

and the corresponding update rule for $\boldsymbol{\lambda}$ becomes

$$\begin{aligned} \boldsymbol{\lambda} &\leftarrow \boldsymbol{\lambda} + \mu(\mathbf{g}(\mathbf{x}) + \mathbf{s}^*) && (15)\\ &= \boldsymbol{\lambda} + \mu\big(\mathbf{g}(\mathbf{x}) + \max(-\mathbf{g}(\mathbf{x}) - \boldsymbol{\lambda}/\mu, \mathbf{0})\big) && (16)\\ &= \max(\boldsymbol{\lambda} + \mu\mathbf{g}(\mathbf{x}), 0) && (17)\\ &= \mu\tilde{\mathbf{g}}(\mathbf{x}). && (18)\end{aligned}$$

2 The first derivatives of principal strains

The deformation gradient of a face is the 3×2 matrix

$$\mathbf{F} = \mathbf{X}\boldsymbol{\Delta}\boldsymbol{\beta}, \quad (19)$$

where $\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix}$ is the matrix of vertex positions. Let its singular value decomposition be $\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^T$, with singular values d_1 and d_2 along the diagonal of $\mathbf{D}$. We want to find the derivatives of d_1 and d_2 with respect to $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$.

In index notation, $\mathbf{F}$ is defined by the equation

$$F_{ij} = \sum_{k,l} X_{ik}\Delta_{kl}\beta_{lj}. \quad (20)$$

Knowing that the derivatives of the singular values with respect to the entries of $\mathbf{F}$ are given by

$$\frac{\partial d_k}{\partial F_{ij}} = U_{ik}V_{jk}, \quad (21)$$

we can find the derivatives with respect to the entries of $\mathbf{X}$ as

$$\begin{aligned} \frac{\partial d_k}{\partial X_{ij}} &= \sum_{l,m}\frac{\partial d_k}{\partial F_{lm}}\frac{\partial F_{lm}}{\partial X_{ij}} && (22)\\ &= \sum_{l,m} U_{lk}V_{mk}\left(\frac{\partial}{\partial X_{ij}}\sum_{n,p} X_{ln}\Delta_{np}\beta_{pm}\right) && (23)\\ &= \sum_{m,p} U_{ik}V_{mk}\Delta_{jp}\beta_{pm} && (24)\end{aligned}$$

because $\partial X_{ln}/\partial X_{ij}$ is nonzero only when $i = l$ and $j = n$. The final equation in traditional matrix-vector notation is

$$\begin{aligned} \frac{\partial d_k}{\partial X_{ij}} &= U_{ik}\mathbf{V}_{\cdot k}^T(\boldsymbol{\Delta}_{j\cdot}\boldsymbol{\beta})^T && (25)\\ &= (\boldsymbol{\Delta}_{j\cdot}\boldsymbol{\beta}\mathbf{V}_{\cdot k})U_{ik}. && (26)\end{aligned}$$

As the components of the gradient $\nabla_{\mathbf{x}_j} d_k$ with respect to the position of vertex j are simply $\partial d_k/\partial X_{\cdot j}$, we finally obtain

$$\nabla_{\mathbf{x}_j} d_k = (\boldsymbol{\Delta}_{j\cdot}\boldsymbol{\beta}\mathbf{V}_{\cdot k})\mathbf{U}_{\cdot k}. \quad (27)$$